代数幾何学シンポジウム記録
2011年度 pp.152-152

On the existence of toric crepant resolution of toric hyperquotient singularities in dimention three

Kohei Sato
sato-kohei@ed.tmu.ac.jp
Tokyo Metropolitan University
Supported by JSPS

Abstract: We show an equivalent condition of the existence of a toric crepant resolution of a hyperquotient singularity which is given by quotients of a three-dimensional affine toric terminal singularity by diagonal group actions.

1 Motivation

X: a normal algebraic variety with isolated Gorenstein singularities
G: a finite group acting on X
Our primary question
What conditions are sufficient for existence of crepant resolutions of quotient singularities $(X/G, x)$?

Assumption
X: an affine toric terminal 3-fold, G: a finite group which acts on X diagonally, $(X/G, x)$: an isolated Gorenstein singularity

Theorem 1.1 **(G. K. White, D. Morrison, G. Stevens, V. Danilov and M. Frumkin)** *Let X be an affine toric Q-factorial threefold. Then X is terminal if and only if X is of the type $\frac{1}{r}(a,-a,1)$ where the integer a is coprime to r. In particular, if X is Gorenstein, then X is smooth.*

Theorem 1.2 *Let X be an affine toric non-Q-factorial threefold. Then X has a terminal singularity if and only if $X \cong \mathrm{Spec}(C[x,y,z,w]/(xz-yw))$.*

2 Quotient case

Let X be of the type $\frac{1}{r}(a,-a,1)$. We define $G' \subset GL(3,C)$ as follows where ε_r is a primitive r-th root of 1,.

$$G' := \langle \begin{pmatrix} \varepsilon_r^a & 0 & 0 \\ 0 & \varepsilon_r^{-a} & 0 \\ 0 & 0 & \varepsilon_r \end{pmatrix} \rangle$$

Definition 2.1 *Let $G \subset GL(3,C)$ be a finite diagonal group action on C^3 as $(x_1,x_2,x_3) \mapsto (\varepsilon_k^a x_1, \varepsilon_k^b x_2, \varepsilon_k^c x_3)$ where $\mathrm{GCD}(a,b,c,k)=1$ and $a,b,c \in Z \cap [0,r)$. If G contains G' as a normal subgroup, then G is called a diagonal group action on X.*

Theorem 2.1 **(Y.Ito, D.G.Markushevich, S.S.Roan)** *All three-dimensional Gorenstein quotient singularities possess crepant resolutions.*

Let G be a diagonal group action on X.
Q. Do there exist diagonal group actions on X such that $(X/G, x)$ are isolated Gorenstein quotient singularities?

Theorem 2.2 **(K. Kurano and S. Nishi)** *Let n be an odd prime number. Let G be a finite subgroup of $GL(n,C)$ which does not contain any pseudo-reflections. Assume that the C^n/G is Gorenstein with isolated singularity. Then C^n/G has a cyclic quotient singularity.*

Example 2.1 *$(X,[0])$: a quotient singularity of type $\frac{1}{r}(a,-a,1)$*

$$G'' := \langle \begin{pmatrix} \varepsilon_r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_r & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon_r \end{pmatrix} \rangle$$

$$\begin{tikzcd} C^3 \cong \mathrm{Spec}C[x,y,z] \arrow[d, "\phi_2"'] \arrow[dr, "\phi_1"] & \\ & X \arrow[dl] \\ C^3 \cong \mathrm{Spec}C[x^r,y^r,z^r] & \end{tikzcd} \qquad \begin{tikzcd} C^3 \cong \mathrm{Spec}C[a,b,c] \arrow[d, "\phi_3"'] \\ C^3/G''' \end{tikzcd}$$

ϕ_2: the quotient map by G''
$G''' \subset GL(3,C)$: small finite, $(C^3/G''',0)$: an isolated Gorenstein singularity $\Rightarrow$ ϕ_3 is a cyclic quotient.

$$G''' := \langle \begin{pmatrix} \varepsilon_k^u & 0 & 0 \\ 0 & \varepsilon_k^v & 0 \\ 0 & 0 & \varepsilon_k^w \end{pmatrix} \rangle \text{ where } \mathrm{GCD}(u,v,w,k)=1.$$

We may identify the variables as $x^r = a,\ y^r = b,\ z^r = c$.

$$\begin{tikzcd} C^3 \cong C[x,y,z] \arrow[d, "\phi_2"'] \arrow[dr, "\phi_1"] \arrow[dd, bend right=60, "\phi_4"'] & \\ C^3 \cong C[x^r,y^r,z^r] \arrow[d, "\phi_3"'] & X \arrow[l] \arrow[dl, "\phi_5"] \\ X/G \cong C^3/\acute{G} & \end{tikzcd}$$

ϕ_4: the composition of quotient maps ϕ_2 and ϕ_3 $\Rightarrow$ $^\exists$ the quotient map ϕ_5.

$$\phi_4 \rightsquigarrow \acute{G} = \langle \begin{pmatrix} \varepsilon_r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_r & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon_r \end{pmatrix}, \begin{pmatrix} \varepsilon_{kr}^{ur} & 0 & 0 \\ 0 & \varepsilon_{kr}^{vr} & 0 \\ 0 & 0 & \varepsilon_{kr}^{wr} \end{pmatrix} \rangle$$

Note 2.1 *X: a quotient singularity of the type $\frac{1}{r}(a,-a,1)$ $\Rightarrow$ $^\exists G$: a diagonal group action on X s.t. $(X/G,0)$ is an isolated Gorenstein singularity $\Rightarrow$ For the G, $^\exists$ a crepant resolution.*

3 Hypersurface case

Let X be $\mathrm{Spec}(C[x,y,z,w]/(xz-yw))$.

Definition 3.1 *Let G be a finite diagonal subgroup (i.e., generated by diagonal matrices) of $GL(4,C)$ acting on C^4 as follows where ε_k is a primitive k-th root of 1 and $a,b,c,d \in [0,r) \cap Z$.*

$$(x,y,z,w) \mapsto (\varepsilon_k^a x, \varepsilon_k^b y, \varepsilon_k^c z, \varepsilon_k^d w)$$

If the group G acts on X, then we call G a diagonal group action on X.

Proposition 3.1 *Let G be a diagonal group action on X and g be a generator of G. If the quotient X/G is Gorenstein, then g can be written as*

$$\begin{pmatrix} \varepsilon_k^a & 0 & 0 & 0 \\ 0 & \varepsilon_k^b & 0 & 0 \\ 0 & 0 & \varepsilon_k^{-a} & 0 \\ 0 & 0 & 0 & \varepsilon_k^{-b} \end{pmatrix}.$$

4 Main result

Theorem 4.1 *Let G be a diagonal group action on $\mathrm{Spec}(C[x,y,z,w]/(xz-yw))$ and $(\mathrm{Spec}(C[x,y,z,w]/(xz-yw))/G, 0)$ be an isolated Gorenstein singularity. There exists a toric crepant resolution of $\mathrm{Spec}(C[x,y,z,w]/(xz-yw))/G$ if and only if G is one of the following;*

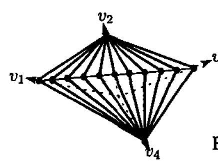
$$\langle \begin{pmatrix} \varepsilon_k^a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \varepsilon_k^{-a} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rangle \text{ or } \langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \varepsilon_k^b & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \varepsilon_k^{-b} \end{pmatrix} \rangle.$$

Moreover

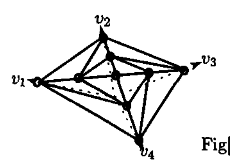
$$\chi(\widetilde{X/G}) = 2|G|$$

where $\widetilde{X/G}$ is a crepant resolution of $\mathrm{Spec}(C[x,y,z,w]/(xz-yw))/G$.

Example 4.1



Fig[1]



Fig[2]

Fig [1]: $G = \langle \begin{pmatrix} \varepsilon_{10} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \varepsilon_{10}^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rangle$ Fig [2]: $G = \langle \begin{pmatrix} \varepsilon_{12}^3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \varepsilon_{12}^{-3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \varepsilon_{12}^4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \varepsilon_{12}^{-4} \end{pmatrix} \rangle$

Fig[1] is a toric crepant resolution. Fig[2] is NOT a crepant resolutioin.